M208/E
2018 exam revised for refreshed module

Module Examination 2018

Pure Mathematics

Thursday 7 June 2018 **10.00 am – 1.00 pm**

Time allowed: 3 hours

This revised paper is intended to give you an idea of how the June 2018 M208 exam paper might have looked had it been prepared for the refreshed module (first presented in 18J). The old questions have been used as far as possible, but put into the new format for the exam paper with some changes to notation and wording where appropriate. The new paper has only 10 questions for Part 1 instead of 12 and the two questions that have been removed are given after the end of the exam paper.

Question 4 is new and replaces a question on material that is no longer studied in the linear algebra units.

Question 10(b) is new, to justify this question being awarded 8 marks instead of 6. All of the other 8 mark questions were originally awarded 6 marks but it is appropriate to award them 8 marks in the new exam format.

Question 12(e) is new as the end of the original question contained material that is now in Group theory 2 instead of Group theory 1.

The rubric, normally on this front page, follows on the next page. The wording of the instructions is as used from June 2020 onwards.

There are **three sections** in this examination.

In **Section 1** you should **attempt all 10 questions**
This section is worth 70% of the total mark.

In **Section 2** you should **attempt 1 out of the 3 questions**. Each question is worth 15% of the total mark.

In **Section 3** you should **attempt 1 out of the 2 questions**. Each question is worth 15% of the total mark.

Include all your working, as some marks are awarded for this.
Write your answers in the answer book(s) provided in **pen**, though you may draw diagrams in pencil.
Start your answer to each question on a new page, clearly indicating the number of the question.
Crossed out work will not be marked.

Calculators are NOT permitted in this examination.

Section 1

You should **attempt all questions**. **This section is worth 70%.**

Question 1 – 5 marks

Sketch the graph of the function f defined by

$$f(x) = \begin{cases} 1 - e^x, & x < 0, \\ \sin\left(\dfrac{\pi x}{2}\right), & x \geq 0, \end{cases}$$

indicating clearly the main features. *[5]*

Question 2 – 5 marks

Let $\sim$ be the relation defined on $\mathbb{Z}$ by $x \sim y$ if $x + y$ is divisible by 2.

Prove that $\sim$ is an equivalence relation. *[5]*

Linear algebra questions (Book C)

Question 3 – 6 marks

This question concerns the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 6 \end{pmatrix}.$$

(a) Determine the eigenvalues of $\mathbf{A}$. *[2]*

(b) For each eigenvalue of $\mathbf{A}$, find a corresponding eigenvector. *[3]*

(c) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$. *[1]*

Question 4 – 6 marks

Find the matrix of the linear transformation

$$\begin{aligned} t\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (2x + y, 2x - y) \end{aligned}$$

with respect to each of the following.

(a) The standard basis for both the domain and the codomain. *[1]*

(b) The basis $\{(1, -3), (-1, 4)\}$ for the domain and the standard basis for the codomain. *[2]*

(c) The basis $\{(1, -3), (-1, 4)\}$ for both the domain and the codomain. *[3]*

Group theory questions (Books B and E)

Question 5 – 8 marks

The permutations $p = (1\,2)(3\,4\,5\,6)$ and $q = (1\,3)(2\,6\,5)$ are elements of S_6.

(a) Find each of $p \circ q$ and $q \circ p$ as a permutation in cycle form and state the order of each of $p \circ q$ and $q \circ p$. *[2]*

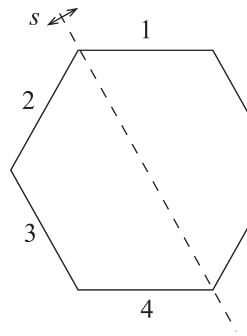
(b) Express each of p and q as a composite of transpositions and hence determine the parity of each of p and q. *[2]*

(c) Determine the cyclic subgroup H of S_6 generated by p, giving the elements of H in cycle form. *[2]*

(d) Explain why $t = (1\,2\,3)(4\,5)$ is conjugate to one of the elements p or q in S_6, and determine two elements in S_6 which conjugate t to your chosen element. *[2]*

Question 6 – 8 marks

This question concerns the group G of symmetries of the regular hexagon, where the edges are labelled as shown below.



Let r be the symmetry given in cycle form as $(1\,3\,5)(2\,4\,6)$, and s be the reflection in the axis shown.

(a) Describe r geometrically and write down s in cycle form, using the labelling of the edges shown above. *[2]*

(b) Express the conjugate symmetry $r \circ s \circ r^{-1}$ in cycle form and describe this symmetry geometrically. *[2]*

(c) Find the conjugacy class of G containing the element r, expressing the elements in cycle form. *[2]*

(d) Using cycle form, write down a subgroup of G of order 3 and explain why it is normal in G. *[2]*

Question 7 – 8 marks

The set

$$G = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\}$$

is a group under matrix multiplication.

This question concerns the mapping ϕ defined by

$$\begin{aligned} \phi\colon G &\longrightarrow (\mathbb{R}^*, \times) \\ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} &\longmapsto a^2. \end{aligned}$$

(a) Prove that ϕ is a homomorphism. [2]

(b) Determine the kernel and image of ϕ. [5]

(c) State a standard group isomorphic to the quotient group $G/\operatorname{Ker}\phi$, justifying your answer briefly. [1]

Analysis questions (Books D and F)

Question 8 – 8 marks

Determine whether each of the following sequences (a_n) converges or diverges, naming any result or rule that you use. If a sequence does converge, then find its limit.

(a) $a_n = \dfrac{2^n - n^4 + 4(n!)}{4^n + n^3 + 2(n!)}$ [3]

(b) $a_n = \dfrac{4^n + 3n^2}{3^n + 2n^3}$ [5]

Question 9 – 8 marks

Determine whether each of the following series converges or diverges, naming any result or test that you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^4 + 4n^2}{4n^3 + 3n^4}$ [3]

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^3}{2n^5 - n + 4}$ [5]

Question 10 – 8 marks

(a) Determine the interval of convergence of the following power series.

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{(n+2)(-3)^n} \qquad [6]$$

(b) State the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n(x+1)^{n-1}}{(n+2)(-3)^n},$$

and give a brief reason for your answer. [2]

Section 2

You should attempt **one question**. If you attempt more, the score from your best question will count.

Each question is worth 15%.

Question 11

Let t be the linear transformation

$$\begin{aligned} t : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (x + 2y + z,\ x + 5z,\ -x - 3y + z). \end{aligned}$$

(a) Find $\operatorname{Ker} t$, and state its dimension. *[5]*

(b) Find a basis for $\operatorname{Im} t$. *[3]*

(c) Describe $\operatorname{Im} t$ geometrically, and find an equation for it. *[3]*

(d) Hence, or otherwise, for each of the following systems of linear equations, determine how many solutions it has.

(i)
$$\begin{aligned} x + 2y + z &= 2 \\ x \phantom{{}+2y} + 5z &= 2 \\ -x - 3y + z &= 7 \end{aligned}$$

(ii)
$$\begin{aligned} x + 2y + z &= 3 \\ x \phantom{{}+2y} + 5z &= 1 \\ -x - 3y + z &= -4 \end{aligned}$$

[4]

Question 12

The set $G = \{1, 2, 4, 8, 9, 13, 15, 16\}$ is a group under multiplication modulo 17. The group table is shown below.

$\times_{17}$	1	2	4	8	9	13	15	16
1	1	2	4	8	9	13	15	16
2	2	4	8	16	1	9	13	15
4	4	8	16	15	2	1	9	13
8	8	16	15	13	4	2	1	9
9	9	1	2	4	13	15	16	8
13	13	9	1	2	15	16	8	4
15	15	13	9	1	16	8	4	2
16	16	15	13	9	8	4	2	1

(a) For each element g in G, write down the cyclic subgroup $\langle g \rangle$ of G generated by g, and write down the order of g. *[4]*

(b) Is G cyclic? If so, write down all the generators of G. If not, explain why not. *[2]*

(c) Write down an isomorphism from $(G, \times_{17})$ to $(\mathbb{Z}_8, +_8)$ that maps 2 to 1. Show clearly which element of G is mapped to which element of $\mathbb{Z}_8$. *[2]*

(d) Write down all the subgroups of G, giving each subgroup only once and justifying your answer. *[2]*

(e) The set $H = \{1, 6, 8, 13, 22, 27, 29, 34\}$ is a group under multiplication modulo 35. The group table is shown below.

$\times_{35}$	1	6	8	13	22	27	29	34
1	1	6	8	13	22	27	29	34
6	6	1	13	8	27	22	34	29
8	8	13	29	34	1	6	22	27
13	13	8	34	29	6	1	27	22
22	22	27	1	6	29	34	8	13
27	27	22	6	1	34	29	13	8
29	29	34	22	27	8	13	1	6
34	34	29	27	22	13	8	6	1

(i) Show that $(H, \times_{35})$ is not isomorphic to $(G, \times_{17})$.

(ii) Construct a permutation group that is isomorphic to $(H, \times_{35})$, writing down an isomorphism between the two groups. *[5]*

Question 13

(a) Determine whether or not each of the following functions f is continuous at 0.

(i) $f(x) = \begin{cases} e^x - 1, & x \leq 0, \\ 2x^3, & x > 0. \end{cases}$

(ii) $f(x) = \begin{cases} e^x - 2, & x \leq 0, \\ 2x^3, & x > 0. \end{cases}$

(iii) $f(x) = \begin{cases} x^2 \sin^2\left(\dfrac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$ [*10*]

(b) Prove that the polynomial

$$p(x) = x^3 - 2x + \tfrac{1}{2}$$

has exactly three zeros in $\mathbb{R}$. [*5*]

Section 3

You should attempt **one question**. If you attempt more, the score from your better question will count.

Each question is worth 15%.

Question 14

This question concerns the group $(\mathbb{R}^*, \times)$ and the plane $\mathbb{R}^2$.

(a) Show that the following equation defines a group action of $\mathbb{R}^*$ on $\mathbb{R}^2$:

$$r \wedge (x, y) = (rx, y).$$ *[4]*

The remainder of the question refers to this group action.

(b) (i) Find the orbit of each of

$$(0, 1);\ (1, 0);\ (1, -2).$$

(ii) Give a geometric description of all the orbits of the action. *[6]*

(c) Find the stabiliser of each of

$$(0, 1);\ (1, -2).$$ *[3]*

(d) Find $\operatorname{Fix} 2$. *[2]*

Question 15

(a) The function f is defined on the interval $[-2, 2]$ by

$$f(x) = \begin{cases} 0, & x = -2, \\ 1 - x, & -2 < x < 1, \\ x, & 1 \le x \le 2. \end{cases}$$

(i) Sketch the graph of f.

(ii) Determine the values of the Riemann sums $L(f, P)$ and $U(f, P)$ for the partition P of $[-2, 2]$ where

$$P = \{[-2, -1], [-1, \tfrac{1}{2}], [\tfrac{1}{2}, 1], [1, 2]\}.$$ *[8]*

(b) Let

$$I_n = \int_0^{\frac{1}{2}} e^{2x}(2x + 3)^n \, dx, \quad n = 0, 1, 2, \ldots.$$

(i) Evaluate I_0.

(ii) Prove that, for $n \ge 1$,

$$I_n = \frac{4^n e - 3^n}{2} - nI_{n-1}.$$

(iii) Hence determine the values of I_1 and I_2. *[7]*

[**END OF QUESTION PAPER**]

The following are questions from the 2018 paper that could alternatively have been included.

Alternative Question 5 – 8 marks

The Cayley table of a set G with the binary operation $\circ$ is shown below.

$\circ$	p	q	r	s
p	r	s	p	q
q	s	p	q	r
r	p	q	r	s
s	q	r	s	p

(a) Show that G is a group and state its identity element. (You can assume associativity.) *[3]*

(b) Show that G is isomorphic to $\mathbb{Z}_4$. *[2]*

(c) Write down all the subgroups of G. *[2]*

(d) Explain why G is not isomorphic to a subgroup of S_3. *[1]*

Alternative Question 10 – 8 marks

A question similar to this could have been included, with a limit requiring two applications of l'Hôpital's Rule instead of one, in order to make the question worth 8 marks.

Prove that the following limit exists, and determine its value.

$$\lim_{x \to 0} \frac{x^3 + x^2 - 3x}{e^{3x} + e^{-2x} - 2}$$ *[6]*

[END OF QUESTION PAPER]